Quantum Topology lecture 16 (April 2, 2024)
Tuesday

- <u>Quantum Teichmüller space</u> [Chekhov-Fock (1999)], [Kashaev (1998)]

Let Σ be an oriented, punctured surface, possibly with boundary.
Assume that each component of $\partial\Sigma$ contains at least 1 puncture,

and that $\chi(\Sigma) < \frac{1}{2}(\underbrace{\#\text{ of components of }\partial\Sigma}_{=d})$.

(i.e. we assume that Σ admits an ideal triangulation).

An ideal triangulation λ has $n = -3\chi(\Sigma) + 2d$ edges,

and $m = -2\chi(\Sigma) + d$ faces.

Let $\{\lambda_1, \cdots, \lambda_n\}$ be the set of edges

and $\{T_1, \cdots, T_m\}$ the set of faces (triangles).

For each face $T_j \in \{T_1, \cdots, T_m\}$,

the <u>triangle algebra</u> $\mathcal{T}^q_{T_j}$ is generated by $X_{j_1}^{\pm 1}, X_{j_2}^{\pm 1}, X_{j_3}^{\pm 1}$

with relations
$$\begin{cases} X_{j_1}X_{j_2} = q^2 X_{j_2}X_{j_1}, \\ X_{j_2}X_{j_3} = q^2 X_{j_3}X_{j_2}, \\ X_{j_3}X_{j_1} = q^2 X_{j_1}X_{j_3}. \end{cases}$$

2/

The <u>Chekhov-Fock algebra $\mathcal{T}_\lambda^q$</u> is the subalgebra of

$\bigotimes_{1\le j\le m} \mathcal{T}_{T_j}^q$ generated by $X_i^{\pm 1}$, $1\le i\le n$,

where $X_i := X_{ja}\otimes X_{kb}$ if λ_i separates two distinct faces T_j and T_k

,

and $X_i := q^{-1}X_{ja}X_{jb} = q X_{jb}X_{ja}$ if λ_i corresponds to two sides of the same face T_j.

<u>Note</u>: X_i's are quantized version of Thurston's shear coordinates.

Note, $X_i X_j = q^{2\sigma_{ij}} X_j X_i$

where $\sigma_{ij} = a_{ij} - a_{ji} \in \{0, \pm1, \pm2\}$,

$a_{ij} :=$ # of angular sectors

3/

Let $\hat{\mathcal{T}}^q_\lambda$ be the fractional division algebra of $\mathcal{T}^q_\lambda$.

i.e. it consists of non-commutative rational functions in variables $X_1, \cdots, X_n$.

<u>Thm</u> (Chekhov–Fock, Kashaev)

There exists a family of algebra isomorphisms

$$\Phi^q_{\lambda,\lambda'} : \hat{\mathcal{T}}^q_{\lambda'} \to \hat{\mathcal{T}}^q_\lambda$$

defined for any two ideal triangulations λ, λ'

such that

$$\Phi^q_{\lambda\lambda''} = \Phi^q_{\lambda\lambda'} \circ \Phi^q_{\lambda'\lambda''} \quad \text{for any triple } \lambda, \lambda', \lambda''.$$

<u>Def</u> The <u>quantum Teichmüller space $\hat{\mathcal{T}}^q_\Sigma$</u>

is the quotient $\displaystyle \bigsqcup_\lambda \hat{\mathcal{T}}^q_\lambda \Big/ \sim$

given by coordinate change isomorphisms

<u>Rmk</u> Teichmüller space is a cluster variety, with cluster charts associated to each triangulation.

Quantum Teichmüller space is the corresponding quantum cluster variety.

- Quantum trace map

So far, we've encountered two quantizations of character varieties of surfaces:

$$\begin{array}{ccc} \chi^{SL_2(\mathbb{C})}(\Sigma) & \underset{A=-1}{\leftsquigarrow} & Sk_A^{sl_2}(\Sigma) \\ \cup & & \\ T(\Sigma) \subset \chi^{SL_2(\mathbb{R})}(\Sigma) & \underset{q=1}{\leftsquigarrow} & \hat{T}^q_\Sigma \end{array}$$

Q. How are these two quantizations related?

Classically, there is a "trace map" relating coordinates on $\chi^{SL_2(\mathbb{C})}(\Sigma)$ to those of $T(\Sigma)$:

Suppose we're given a positive weight $X_i \in \mathbb{R}_+$ for each interior edge λ_i of the ideal triangulation.
Then we can associate a group homomorphism $r: \pi_1(\Sigma) \to PSL_2(\mathbb{R})$ as follows:

(1) Lift the ideal triangulation λ to
an ideal triangulation $\tilde{\lambda}$ of the universal cover $\tilde{\Sigma}$

(2) Construct an orientation-preserving immersion

$$\tilde{f} : \tilde{S} \to \mathbb{H}^2.$$

such that each face $\tilde{T}$ of $\tilde{\lambda}$ is mapped to an ideal hyperbolic triangle ,

and if $\tilde{T}$ and $\tilde{T}'$ meet along an edge $\tilde{\lambda}_i$,
then $\tilde{f}(\tilde{T}')$ is obtained from $\tilde{f}(\tilde{T})$ by
performing a hyperbolic reflection across $\tilde{f}(\tilde{\lambda}_i)$,
followed by a hyperbolic translation of $\log X_i$ along the same geodesic,
(where we orient $\tilde{f}(\tilde{\lambda}_i)$ by the boundary orientation of $\tilde{T}$).

In this way, we get a group homomorphism

$$r : \pi_1(\Sigma) \longrightarrow PSL_2(\mathbb{R}),$$

unique up to conjugation by an element of $PSL_2(\mathbb{R})$.

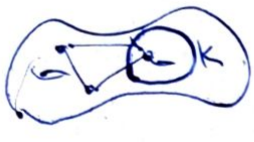
6/

For a group homomorphism $r : \pi_1(\Sigma) \to PSL_2(\mathbb{R})$

and a closed immersed curve $K \subset \Sigma$,

we get a natural lift $\hat{r}(K) \in SL_2(\mathbb{R})$ of $r(K) \in PSL_2(\mathbb{R})$

(in fact to the universal cover of $SL_2(\mathbb{R})$)



$\rightsquigarrow$

$\tilde{K} : [0,1] \to \tilde{\Sigma}$

We're interested in $\mathrm{Tr}\, \hat{r}(K)$.

If K is a small circle bounding a disk embedded in Σ,

then $\hat{r}(K) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\mathrm{Tr}\, \hat{r}(K) = -2$.

If $r : \pi_1(\Sigma) \to PSL_2(\mathbb{R})$ is constructed from shear parameters,

then $\hat{r}(K)$ can be explicitly computed:

 $\rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

 $\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

 $\rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $-I$ for every full turn.

 $\rightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$\begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & X^{-\frac{1}{2}} \end{pmatrix}$

<u>Lemma</u>: Up to conjugation by an element of $SL_2(\mathbb{R})$,

$$\hat{r}(K) = S(X_{i_1}) M_1 \, S(X_{i_2}) M_2 \cdots S(X_{i_k}) M_k$$

7/

~~Let's assign~~

A *state* is an assignment of a sign $s_1, s_2, \cdots, s_k \in \{\pm\}$ to each point where K crosses an edge λ_{i_j} of λ, in this order.

For $j = 1, \cdots, k$, write the matrix M_j defined above as

$$M_j = \begin{pmatrix} m_j^{++} & m_j^{+-} \\ m_j^{-+} & m_j^{--} \end{pmatrix}, \quad \text{with } m_j^{\pm\pm} \in \{0, \pm 1\}.$$

Then we have

$$\mathrm{Tr}\,\hat{r}(K) = \sum_s m_1^{s_1 s_2} m_2^{s_2 s_3} \cdots m_k^{s_k s_1} X_{i_1}^{\frac{s_1}{2}} X_{i_2}^{\frac{s_2}{2}} \cdots X_{i_k}^{\frac{s_k}{2}},$$

where the sum is over all possible states s for K and λ.

This is a formula relating the coordinates given by trace functions to the shear coordinates.

Note: We need square-roots of shear coordinates.

This motivates us to consider square-root Chekhov-Fock algebra $\mathcal{Z}_\lambda^\omega$ (generated by formal square-roots),

and a relative version of skein algebra ("stated skein algebra")

<u>Thm</u> ([Bonahon-Wong (2010)])

There is an algebra homomorphism

$\omega = q_{BW}^{\frac{1}{4}}$

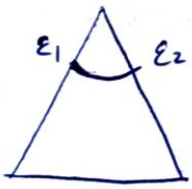
consisting of Laurent polynomials in square roots of quantized shear parameters.

$$\mathrm{Tr}_\lambda^\omega : \mathrm{SkAlg}^{sl_2}_{A=\omega^{-2}}(\Sigma) \to \mathcal{Z}_\lambda^\omega$$

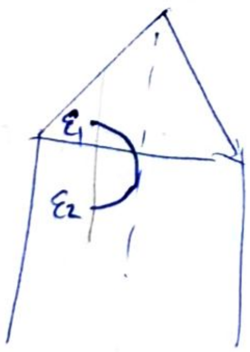
which becomes the classical trace map when $\omega = 1$.

This gives a precise connection between the two quantizations of the $SL_2(\mathbb{C})$-character variety of Σ.

Roughly,



$$\mapsto \begin{cases} 0 & \text{if } (\varepsilon_1, \varepsilon_2) = (-,+) \\ [Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] & \text{otherwise} \end{cases},$$



$$\mapsto \begin{cases} 0 & \text{if } \varepsilon_1 \neq \varepsilon_2 \\ (\text{some scalar}) & \text{otherwise} \end{cases}$$